COHOMOLOGY AND PROJECTIVITY OF MODULES FOR FINITE GROUP SCHEMES

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1. Introduction

Let G be a finite group scheme over a field k, that is, an affine group scheme whose coordinate ring $k[G]$ is finite dimensional. The dual algebra $k[G]^* \equiv \mathrm{Hom}_k(k[G], k)$ is then a finite dimensional cocommutative Hopf algebra. Indeed, there is an equivalence of categories between finite group schemes and finite dimensional cocommutative Hopf algebras (cf. [19]). Further the representation theory of G is equivalent to that of $k[G]^*$. Many familiar objects can be considered in this context. For example, any finite group G can be considered as a finite group scheme. In this case, the algebra $k[G]^*$ is simply the group algebra kG. Over a field of characteristic $p > 0$, the restricted enveloping algebra $u(\mathfrak{g})$ of a p-restricted Lie algebra $\mathfrak{g}$ is a finite dimensional cocommutative Hopf algebra. Also, the mod-p Steenrod algebra is graded cocommutative so that some finite dimensional Hopf subalgebras are such algebras.

Over the past thirty years, there has been extensive study of the modular representation theory (i.e., over a field of positive characteristic $p > 0$) of such algebras, particularly in regards to understanding cohomology and determining projectivity of modules. This paper is primarily interested in the following two questions:

Questions 1.1. Let G be a finite group scheme G over a field k of characteristic $p > 0$, and let M be a rational G-module.

(a) Does there exist a family of subgroup schemes of G which detects whether M is projective?
(b) Does there exist a family of subgroup schemes of G which detects whether a cohomology class $z \in \mathrm{Ext}^n_G(M, M)$ (for M finite dimensional) is nilpotent?

It is shown here that when the connected component of G is unipotent there is a family of subgroup schemes (with simple structure) that provides an affirmative answer to both questions. These are referred to as *elementary* group schemes.

Definition 1.2. Given a field k of characteristic $p > 0$ and a pair of non-negative integers r and s, define the *elementary* group scheme $\mathcal{E}_{r,s}$ to be the product $\mathbb{G}_{a(r)} \times E_s$ over k, where $\mathbb{G}_{a(r)}$ denotes the rth Frobenius kernel of the additive group scheme $\mathbb{G}_a$, and E_s is an elementary abelian p-group of rank s (considered as a finite group scheme). The groups $\mathbb{G}_{a(0)}$ and E_0 are identified with the trivial group.

The precise statements of the main results are Theorems 6.1 and 8.1. Some related detection results are deduced in Sections 7 and 8 as well.

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This work has its roots in work on finite groups by J.-P. Serre [28], D. Quillen [24], L. Chouinard [8], J. Alperin and L. Evens [1], G. Avrunin and L. Scott [2], and J. Carlson [7] among others. For a finite group, the family of elementary abelian subgroups detects projectivity and nilpotence. Specifically, we remind the reader of two fundamental theorems which provide a model for other results.

Theorem 1.3 (Chouinard [8]). *Let k be a field of characteristic $p > 0$, G be a finite group, and M be a kG-module. Then M is projective as a kG-module if and only if it is projective as a kE-module for every elementary abelian subgroup $E \subset G$.*

Theorem 1.4 (Quillen [24], Carlson [7]). *Let k be a field of characteristic $p > 0$, G be a finite group, and M be a finite dimensional kG-module. Then an element $z \in \operatorname{Ext}^*_{kG}(M, M)$ is nilpotent if and only if the image of z under the the natural restriction map $\operatorname{Ext}^*_{kG}(M, M) \to \operatorname{Ext}^*_{kE}(M, M)$ is nilpotent for every elementary abelian subgroup $E \subset G$.*

More recent forerunners of this work are the analogous results which have been obtained for restricted Lie algebras or more generally for infinitesimal group schemes. An infinitesimal group scheme H is one for which the coordinate algebra $k[H]$ is finite dimensional and local. One of the most important examples of such is the set of Frobenius kernels $\{G_{(r)}\}$ of an affine group scheme G, where $G_{(r)}$ is the kernel of the rth iterate of the Frobenius map on G. Any restricted Lie algebra $\mathfrak{g}$ corresponds to a certain group scheme $G_{(1)}$, the first Frobenius kernel of some affine group scheme (cf. [19]). In work of E. Friedlander, A. Sulin, and the author [30], [3], it was shown that the family of subgroup schemes $\{\mathbb{G}_{a(r)}\}$ plays the role of elementary abelian subgroups in detecting projectivity and nilpotence of cohomology for infinitesimal group schemes. Specifically, Theorems 2.5 and 4.1 of [30] are analogues of Theorem 1.4 for detecting nilpotent cohomology classes, and Proposition 7.6 of [30] is an analogue of Chouinard's theorem (Theorem 1.3) about detecting projectivity. However, the latter result holds only for finite dimensional modules, whereas Chouinard's theorem holds for arbitrary modules. In [3], ideas based on Chouinard's original arguments were used to show that the family of subgroup schemes $\{\mathbb{G}_{a(r)}\}$ does detect projectivity of arbitrary modules for infinitesimal *unipotent* group schemes. It remains an open problem to show this for arbitrary infinitesimal group schemes.

Given the results for finite groups and infinitesimal group schemes, one is naturally led to ask the questions in 1.1. In other words, is there a "nice" general family of subgroup schemes which detects projectivity and nilpotence?

The search for an answer begins by noticing a similarity between the fundamentally different notions of an elmentary abelian p-group E_r of rank r and the Frobenius kernel $\mathbb{G}_{a(r)}$. Indeed, their corresponding Hopf algebras $k[E_r]^* = kE_r$ and $k[\mathbb{G}_{a(r)}]^*$ are both isomorphic as algebras to a truncated polynomial algebra $A_r = k[x_1, x_2, \ldots, x_r]/(x_1^p, x_2^p, \ldots, x_r^p)$ (although their coalgebra structures differ). This suggests considering a general family of group schemes whose coordinate algebras have such structure, and leads to the above definition (1.2) of elementary group schemes.

It is shown here that for finite group schemes whose (infinitesimal) connected component at the identity, G_0, is *unipotent* the family of elementary group schemes $\{\mathcal{E}_{r,s}\}_{r\geq 0, s\geq 0}$ plays the general role of elementary abelian groups. More precisely,

the first main result of this paper, Theorem 6.1, is a generalization of Theorem 1.4 above and Theorem 2.5 of [30] on detecting nilpotent cohomology classes. And the second main result, Theorem 8.1, is a generalization of Theorem 1.3 above and the Theorem of [3] on detecting projectivity of modules.

A group scheme G is unipotent if it admits an embedding as a closed subgroup of U_n for some n, where U_n is the subgroup of invertible strictly upper triangular matrices in GL_n (the group scheme of all invertible $n \times n$ matrices). Unipotent groups are generalizations of p-groups in that they admit only a single simple module, the trivial module k. (Indeed, that is sometimes taken as the definition of a unipotent group scheme. And if the reader prefers a Hopf algebra perspective, then unipotent means that we are considering Hopf algebras which admit a single simple module.)

The connected component of G comes into play via a decomposition of a finite group scheme into an infinitesimal part and a finite group part. More precisely, if G is a finite group scheme over k and all of its points are k-rational, then G may be identified as a semi-direct product $G = G_0 \rtimes \pi$, where $\pi = G(k)$ is the finite group of k-points of G which acts on G_0 via group scheme automorphisms (cf. [17], [31]). In the proofs of the main results, a certain injection in cohomology (Lemma 5.4) leads to an easy reduction to the case that π is a p-group. When π is a p-group and G_0 is unipotent, the group G is unipotent.

Unipotent groups have other nice properties which will also be reviewed in Section 5. Prior to that, some necessary cohomological properties of elementary group schemes are developed in Sections 2 and 3, with the key result being Proposition 3.6. This is a characterization of the cohomology of elementary group schemes that generalizes Serre's characterization of the cohomology of elementary abelian groups (cf. Proposition 3.4). The proofs of the main results, rely on homomorphisms of the form $G \twoheadrightarrow \mathbb{G}_{a(1)}$ and $G \twoheadrightarrow \mathbb{Z}/p$. Some cohomological properties related to such homomorphisms are also developed in Sections 4 and 5.

Theorem 6.1 (whose statement and proof make up the content of Section 6) has several almost immediate consequences which are presented in Section 7. One of these, Corollary 7.2, is a generalization of Theorem 1.3 about detecting projectivity of *finite dimensional* modules. Indeed, for finite dimensional modules, Chouinard's theorem (Theorem 1.3) follows via a general argument from Theorem 1.4. The key fact is that for a finite dimensional cocommutative Hopf algebra A over k and a finite dimensional A-module M, $\mathrm{Ext}^*_A(M,M)$ is finitely generated as a module over (the Noetherian ring) $H^*(A,k) = \mathrm{Ext}^*_A(k,k)$ (cf. [17]). The bulk of Section 8 consists of the proof of Theorem 8.1 which gives the detection of projectivity for arbitrary modules. In the final result of the paper (Corollary 8.6), Dade's lemma [11] is used to derive an alternate detection result.

Having outlined the contents of the paper, we note some additional motivation for answering the questions in 1.1. First, Theorem 1.4 is a key component in identifying the *cohomological support varieties* of finite groups. For any finite dimensional cocommutative Hopf algebra A over a field k, the cohomology ring $H^*(A,k) = \mathrm{Ext}^*_A(k,k)$ is graded commutative so that the even dimensional portion $H^{2*}(A,k)$ is a commutative ring. The cohomological variety (or properly scheme) of A is defined to be the prime ideal spectrum $\mathrm{Spec}\, H^{2*}(A,k)$. Further, for any finite dimensional A-module

M, consider the kernel $J_A(M)$ of the map

$$\mathrm{Ext}_A^{2*}(k,k) \xrightarrow{\otimes M} \mathrm{Ext}_A^*(M,M).$$

This is a homogeneous ideal which defines a homogeneous subscheme of Spec $H^{2*}(A,k)$ called the support variety of M. These definitions were originally made in terms of the maximal ideal spectrum (over algebraically closed fields) in which case they are honest varieties. Within the literature, it has been noted that it is often more convenient to work instead with prime ideals, while maintaining the traditional terminology.

This study of cohomology via varieties was extended to restricted Lie algebras by E. Friedlander and B. Parshall [15], [16], as well as by J. Jantzen [20], and later to arbitrary infinitesimal group schemes by E. Friedlander, A. Suslin, and the author [29, 30]. In that work, the detection properties of the family of subgroup schemes $\{\mathbb{G}_{a(r)}\}$ was crucial for identifying the varieties. This suggests that a substantial theory of cohomological varieties for arbitrary finite group schemes requires a family of subgroup schemes (with simple cohomological structure) which detects nilpotence.

Further, D. Benson, J. Carlson, and J. Rickard [5, 6] have recently developed a theory of varieties for arbitrary modules (i.e., including infinite dimensional modules) which seems to be fruitful for studying finite dimensional modules. A family of subgroup schemes which detects both nilpotence in cohomology and projectivity of arbitrary modules is important for an attempt to extend these ideas to more general finite group schemes or Hopf algebras, because some of their results depend on knowing that Theorem 1.3 holds for arbitrary modules. Indeed, the recent work of M. Hovey and J. Palmieri [18] extends the definitions of these varieties to some more general finite dimensional graded cocommutative Hopf algebras.

While the work here is from a group scheme perspective, these results could also be stated in terms of finite dimensional cocommutative Hopf algebras. From the latter perspective, the results say that nilpotence in cohomology and projectivity of modules can be detected by certain *elementary* Hopf subalgebras, that is, Hopf subalgebras which have the algebra structure of a truncated polynomial algebra $k[x_1, x_2, \ldots, x_n]/(x_1^p, x_2^p, \ldots, x_n^p)$. We leave the precise formulations to the interested reader, but point out that others have studied these questions from a Hopf algebra perspective, in fact, from the more general perspective of graded Hopf algebras.

A few examples are work of C. Wilkerson [32], J. Palmieri [26], D. Nakano and J. Palmieri [23], and M. Hovey and J. Palmieri [18]. The motivation for considering graded algebras comes from topological interests in the Steenrod algebra which is graded. While any Hopf algebra can be trivially graded (by concentrating the algebra in degree 0), a generic graded Hopf algebra may only be *graded* cocommutative and not honestly cocommutative. Such algebras do not correspond to finite group schemes and are not in our realm of consideration. Even if one ignores the grading, one must be careful about issues of whether modules and subalgebras are graded or not. For example, working over graded Hopf algebras, C. Wilkerson [32] showed that elementary Hopf subalgebras did not necessarily detect nilpotence in the cohomology ring of the restricted enveloping algebra of a restricted Lie algebra. However, his example does not contradict our results because his example is for a graded Lie algebra and he only considers graded subalgebras, while Theorem 6.1 would require looking at a larger family of elementary subalgebras.

Wilkerson's work led to the more general notion of *quasi-elementary* Hopf algebras. This family of Hopf algebras was used successfully by J. Palmieri [26] to obtain an analogue of Theorem 1.4 and many other results in work with D. Nakano [23] and M. Hovey [18] for graded (connected) finite dimensional cocommutative Hopf algebras. While some of these results would apply to our situation, they remain somewhat unsatisfying because, while quasi-elementary Hopf algebras can be defined cohomologically, they have only been concretely described in some cases. For a group algebra, they simply correspond to elementary abelian groups. For a finite dimensional Hopf subalgebra of the mod-2 Steenrod algebra, which is honestly cocommutative, the quasi-elementary subalgebras are in fact elementary. In that case, Theorem 6.1 is consistent with [26] in claiming that nilpotence can be detected by elementary Hopf subalgebras.

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2. Cohomology of Elementary Group Schemes

For the remainder of the paper, k will denote a field of characteristic $p > 0$ and group schemes will be defined over k (unless otherwise specified). As group schemes of the form $\mathcal{E}_{r,s} = \mathbb{G}_{a(r)} \times E_s$ will play a prominent role in the sequel, we make note of the structure of their cohomology rings. Of course we have

$$H^*(\mathcal{E}_{r,s}, k) = H^*(\mathbb{G}_{a(r)}, k) \otimes H^*(E_s, k).$$

And the cohomology of both $\mathbb{G}_{a(r)}$ and E_s are well known, the first via [10] and the second is a classical result (see for example [4] or [13]). Indeed, their algebra structures are identical when $r = s$, yet it will be important to distinguish the two pieces, and so we use different notation.

Proposition 2.1. *If $p \neq 2$, then $H^*(\mathbb{G}_{a(r)}, k) = k[x_1, x_2, \ldots, x_r] \otimes \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $H^*(E_s, k) = k[z_1, z_2, \ldots, z_s] \otimes \Lambda(y_1, y_2, \ldots, y_s)$, where the x_i and z_i have degree 2 and the λ_i and y_i have degree 1. If $p = 2$, then we simply have $H^*(\mathbb{G}_{a(r)}, k) = k[\lambda_1, \lambda_2, \ldots, \lambda_r]$ and $H^*(E_s, k) = k[y_1, y_2, \ldots, y_s]$ with λ_i and y_i again having degree 1. In the latter case, we set $x_i = \lambda_i^2$ and $z_i = y_i^2$ for each i.*

The cohomology ring of any finite-dimensional cocommutative Hopf algebra admits the action of Steenrod operations $\beta\mathcal{P}^i$ and $\mathcal{P}^i$ (or Sq^i) (cf. [22]). Of essential importance for our proofs will be the action of the Steenrod operations on $H^*(\mathcal{E}_{r,s}, k)$. We summarize in the following Lemma the actions on the above generators, and note that the actions differ somewhat on the two pieces. The action on the E_s generators is well known (cf. [4], [22]). For $\mathbb{G}_{a(r)}$ we refer the reader to [30, Prop. 1.7]. Indeed, in the case of finite groups, the operation $\beta\mathcal{P}^i$ is simply the composition of $\mathcal{P}^i$ with the Bockstein homomorphism β.

Lemma 2.2. *Let r and s be non-negative integers. The action of the Steenrod operations on the generators of $H^*(\mathcal{E}_{r,s}, k)$ as given in Proposition 2.1 are as follows:*

- *For* $p > 2$,
  (a) $\mathcal{P}^j(x_i) = 0 = \mathcal{P}^j(z_i)$, $j > 1$; $\mathcal{P}^j(\lambda_i) = 0 = \mathcal{P}^j(y_i)$, $j \geq 1$;
  $\beta\mathcal{P}^j(x_i) = 0 = \beta\mathcal{P}^j(z_i)$, $j \geq 1$; $\beta\mathcal{P}^j(\lambda_i) = 0 = \beta\mathcal{P}^j(y_i)$, $j \geq 1$.
  (b) $\mathcal{P}^0(x_i) = x_{i+1}$, $\mathcal{P}^0(z_i) = z_i$, $\mathcal{P}^0(\lambda_i) = \lambda_{i+1}$, $\mathcal{P}^0(y_i) = y_i$.
  (c) $\mathcal{P}^1(x_i) = x_i^p$, $\mathcal{P}^1(z_i) = z_i^p$.
  (d) $\beta\mathcal{P}^0(x_i) = 0 = \beta\mathcal{P}^0(z_i)$, $\beta\mathcal{P}^0(\lambda_i) = -x_i$, $\beta\mathcal{P}^0(y_i) = z_i$.
  (e) $\mathcal{P}^m(x_r^\ell) = x_r^{p\ell}$ *if* $m = \ell$; $\mathcal{P}^m(x_r^\ell) = 0$ *if* $m \neq \ell$.
- *For* $p = 2$,
  (a) $Sq^j(x_i) = 0 = Sq^j(z_i)$, $j > 2$; $Sq^j(\lambda_i) = 0 = Sq^j(y_i)$, $j > 1$.
  (b) $Sq^0(x_i) = x_{i+1}$, $Sq^0(z_i) = z_i$, $Sq^0(\lambda_i) = \lambda_{i+1}$, $Sq^0(y_i) = y_i$.
  (c) $Sq^1(x_i) = 0 = Sq^1(z_i)$, $Sq^1(\lambda_i) = x_i$, $Sq^1(y_i) = z_i$.
  (d) $Sq^2(x_i) = x_i^2$, $Sq^2(z_i) = z_i^2$.
  (e) $Sq^m(x_r^\ell) = x_r^{2\ell}$ *if* $m = 2\ell$; $Sq^m(x_r^\ell) = 0$ *if* $m \neq 2\ell$.

3. Serre's Theorem

An essential ingredient in proving both Theorems 1.3 and 1.4 is Serre's cohomological characterization of elementary abelian p-groups:

Theorem 3.1 (Serre [28]). *Let k be a field of characteristic $p > 0$ and G be a finite p-group. If G is not elementary abelian, then there exists a finite family of non-zero elements $v_1, v_2, \ldots, v_m \in H^1(G,k)$ such that the element $u = \prod \beta(v_i) \in H^2(G,k)$ is zero, where β denotes the Bockstein homomorphism.*

For our purposes it is more useful to work with a "dual" statement about the cohomology of elementary abelian groups. We present this in Proposition 3.4 followed by an analogue and a generalization. The proof of Proposition 3.4 is essentially contained in Serre's original proof of Theorem 3.1 (which was stated for the finite field $k = \mathbf{F}_p$). Throughout this section we use the notation of Proposition 2.1. The following fact is a special case of the Corollaire to Proposition (1) of [28].

Proposition 3.2 (Serre [28]). *Let s be a positive integer and $I \subset \mathbf{F}_p[z_1, z_2, \ldots, z_s] \subset H^*(E_s, \mathbf{F}_p)$ be a non-zero homogeneous ideal in $\mathbf{F}_p[z_1, z_2, \ldots, z_s]$ which is stable under the Steenrod operations. Then there exists a finite family $\{u_i\}$ of elements in $\mathbf{F}_p[z_1, z_2, \ldots, z_s]$, each of which is a non-zero linear combination of the $\{z_j\}$, such that the product $\prod u_i \in \mathbf{F}_p[z_1, z_2, \ldots, z_s]$ lies in I.*

Proof. Let $A = \mathbf{F}_p[z_1, z_2, \ldots, z_s]$ and consider the operation $\theta : A \to A$ defined by $\theta = \sum_{i=0}^{\infty} \mathcal{P}^i$ (or the sum of the Sq^i in the case $p = 2$). It follows from the Cartan formula, $\mathcal{P}^\ell(u \cdot v) = \sum_{n=0}^{\ell} \mathcal{P}^n(u) \cdot \mathcal{P}^{\ell-n}(v)$, that θ is an algebra homomorphism. Moreover, from Lemma 2.2, it is the algebra homomorphism defined by $\theta(z_i) = z_i + z_i^p$ for each i. Finally, since I is stable under the Steenrod operations, it is stable under θ, and so the claim follows from the Corollaire to Proposition (1) of [28]. □

The following lemma allows us to extend from the finite field $\mathbf{F}_p$ to an arbitrary field k of characteristic $p > 0$.

Lemma 3.3. *Let k be a field of characteristic $p > 0$ and s be a positive integer. If $I \subset k[z_1, z_2, \ldots, z_s] \subset H^*(E_s, k)$ is a non-zero homogeneous ideal in $k[z_1, z_2, \ldots, z_s]$ which*

is stable under the Steenrod operations, then $I' = I \cap \mathbf{F}_p[z_1, z_2, \ldots, z_s]$ is also non-zero, where the algebra $\mathbf{F}_p[z_1, z_2, \ldots, z_s]$ is naturally embedded in the algebra $k[z_1, z_2, \ldots, z_s]$.

Proof. Let

$$f = \sum_{i=1}^{n} a_i z_1^{j_{i,1}} z_2^{j_{i,2}} \cdots z_s^{j_{i,s}}$$

be a non-zero homogeneous polynomial in I. Choose such an f that has lowest possible degree and for which the number of summands, n, is least. Having made such a choice, it suffices to assume that $a_1 = 1$. From Lemma 2.2 and the Cartan formula ($\mathcal{P}^0(u \cdot v) = \mathcal{P}^0(u) \cdot \mathcal{P}^0(u)$), we see that

$$\mathcal{P}^0(f) = \sum_{i=1}^{n} a_i^p z_1^{j_{i,1}} z_2^{j_{i,2}} \cdots z_s^{j_{i,s}}.$$

Since $a_1 = 1$, the difference

$$\mathcal{P}^0(f) - f = \sum_{i=2}^{n} (a_i^p - a_i) z_1^{j_{i,1}} z_2^{j_{i,2}} \cdots z_s^{j_{i,s}}$$

has fewer summands than f. By assumption $\mathcal{P}^0(f)$ and hence $\mathcal{P}^0(f) - f$ also lie in I. By the minimality of f, $\mathcal{P}^0(f) - f$ must be zero. That is, we must have $a_i^p = a_i$ for each $1 \leq i \leq n$. In other words, f lies in the subalgebra $\mathbf{F}_p[z_1, z_2, \ldots, z_s]$ and so the ideal I' is non-zero. □

Proposition 3.4 (Serre [28]). *Let k be a field of characteristic $p > 0$, s be a positive integer, and $I \subset H^*(E_s, k)$ be a homogeneous ideal which is stable under the Steenrod operations. If I contains a non-zero element of degree two, then there exists a finite family $\{u_i\}$ of elements in $H^2(E_s, k)$, each of which is a non-zero linear combination of the $\{z_j\}$, such that the product $\prod u_i \in H^*(E_s, k)$ lies in I.*

Proof. We have $H^*(E_s, k) = k[z_1, z_2, \ldots, z_s] \otimes \Lambda(y_1, y_2, \ldots, y_s)$. Consider the set $I' = I \cap k[z_1, z_2, \ldots, z_s] \subset k[z_1, z_2, \ldots, z_s]$. This is evidently an ideal in $k[z_1, z_2, \ldots, z_s]$ and moreover homogeneous since I was. If I' is non-zero, by Lemma 3.3, the ideal $I'' = I \cap \mathbf{F}_p[z_1, z_2, \ldots, z_s] \subset \mathbf{F}_p[z_1, z_2, \ldots, z_s]$ is also non-zero as well as homogeneous and Steenrod stable. Applying Proposition 3.2 to I'', the result follows.

Let $z = \sum_{1 \leq i < j \leq s} a_{i,j} y_i y_j + \sum_{1 \leq \ell \leq s} b_\ell z_\ell$ with $a_{i,j}, b_\ell \in k$ be a non-zero degree two element in I. If each $a_{i,j} = 0$, then z is an element of the desired form and we're done. If not, as in the proof of Proposition (4) of [28], apply the operation $\beta \mathcal{P}^1 \beta \mathcal{P}^0$ (or $Sq^1 Sq^2 Sq^1$ in the case $p = 2$) to z. From the Cartan formula and Lemma 2.2, we have $\beta \mathcal{P}^1 \beta \mathcal{P}^0(z) = \sum_{i<j} a_{i,j}^{p^2} (z_i^p z_j - z_i z_j^p)$, which is non-zero. Clearly this lies in $k[z_1, z_2, \ldots, z_s]$ and by the stability assumption it lies in I and hence in I'. □

In the case of infinitesimal group schemes, there is a similar sort of result which was used in [30]. For the reader's convenience, we restate it here.

Proposition 3.5 (Proposition 1.7 of [30]). *Let k be a field of characteristic $p > 0$, r be a positive integer, and $I \subset H^*(\mathbb{G}_{a(r)}, k)$ be a k-submodule stable with respect to the Steenrod operations. If I contains a non-zero element of degree two, then some power of x_r lies in I.*

We now combine these into a statement about $\mathcal{E}_{r,s} = \mathbb{G}_{a(r)} \times E_s$.

Proposition 3.6. *Let k be a field of characteristic $p > 0$, r and s be non-negative integers, and $I \subset H^*(\mathcal{E}_{r,s}, k)$ be a homogeneous ideal which is stable under the Steenrod operations. If I contains a non-zero element of degree two, then there exists an integer m and a finite family $\{u_i\}$ of elements in $H^2(E_s, k) \subset H^2(\mathcal{E}_{r,s}, k)$, each of which is a non-zero linear combination of the $\{z_j\}$, such that the product $x_r^m \prod u_i \in H^*(\mathcal{E}_{r,s}, k)$ lies in I.*

Proof. Consider the ideals $I_1 = I \cap H^*(E_s, k)$ and $I_2 = I \cap H^*(\mathbb{G}_{a(r)}, k)$. These ideals satisfy the structural hypotheses of Propositions 3.4 and 3.5 respectively. By assumption I contains some non-zero element u of degree 2. Generically, u has the following form:

$$u = \sum_{1 \le i < j \le r} a_{i,j} \lambda_i \lambda_j + \sum_{1 \le j \le r} b_j x_j + \sum_{1 \le i \le r, 1 \le j \le s} c_{i,j} \lambda_i y_j + \sum_{1 \le i < j \le s} d_{i,j} y_i y_j + \sum_{1 \le j \le s} e_j z_j$$

for some constants $a_{i,j}, b_j, c_{i,j}, d_{i,j}, e_j \in k$ which are not all zero.

We remind the reader of Lemma 2.2 which lists the actions of the Steenrod operations on the generators. We consider the case that $p > 2$ and simply make note of the appropriate operations in the case $p = 2$. The interested reader may fill in the details of the latter case. From Lemma 2.2(b) and the Cartan formula:

$$\mathcal{P}^\ell(v \cdot w) = \sum_{n=0}^{\ell} \mathcal{P}^n(v) \cdot \mathcal{P}^{\ell-n}(w),$$

we first observe that repeated application of $\mathcal{P}^0$ (resp. Sq^0) to u results in an element of the form $u' = \sum_{i<j} d_{i,j}^{p^\ell} y_i y_j + \sum_j e_j^{p^\ell} z_j$ since the other terms are eventually killed by $\mathcal{P}^0$ (resp. Sq^0). By the assumption that I is stable under the action of the Steenrod operations, u' lies in I and hence in I_1. So if at least one of the $d_{i,j}$ or e_j is non-zero, applying Proposition 3.4 we're done.

Hence we may assume that u has the form

$$u = \sum_{i<j} a_{i,j} \lambda_i \lambda_j + \sum_j b_j x_j + \sum_{i,j} c_{i,j} \lambda_i y_j$$

with not all coefficients being zero. As already noted, repeated application of $\mathcal{P}^0$ (resp. Sq^0) will eventually kill such a u. Stopping at the last point before we get zero, we may further assume that u has the form

$$u = \sum_{i<r} a_{i,r} \lambda_i \lambda_r + b_r x_r + \sum_j c_{r,j} \lambda_r y_j$$

with not all coefficients being zero. If every $c_{r,j} = 0$, then u lies in I_2 and we are done by Proposition 3.5. Hence we may assume that some $c_{r,j}$ is non-zero. Applying $\beta\mathcal{P}^0$ (resp. Sq^1) and using the Cartan formula:

$$\beta\mathcal{P}^\ell(v \cdot w) = \sum_{n=0}^{\ell} \beta\mathcal{P}^n(v) \cdot \mathcal{P}^{\ell-n}(w) + (-1)^{\dim v} \mathcal{P}^n(v) \beta\mathcal{P}^{\ell-n}(w),$$

along with Lemma 2.2(b,d), we get

$$\beta\mathcal{P}^0(u) = \sum_{i<r} a_{i,r}^p \lambda_{i+1} x_r - \sum_j c_{r,j}^p x_r y_j \in I.$$

And further applying $\beta\mathcal{P}^1$ (resp. Sq^3) to this, using Lemma 2.2(a,b,c,d), we get

$$\beta\mathcal{P}^1\beta\mathcal{P}^0(u) = -\sum_{i<r} a_{i,r}^{p^2} x_{i+1} x_r^p - \sum_j c_{r,j}^{p^2} x_r^p z_j \in I.$$

Now we apply $\mathcal{P}^p$ (resp. Sq^4) to this. With the Cartan formula and Lemma 2.2(a,b,e), we get

$$\mathcal{P}^p\beta\mathcal{P}^1\beta\mathcal{P}^0(u) = -\sum_{i<r} a_{i,r}^{p^3} x_{i+2} x_r^{p^2} - \sum_j c_{r,j}^{p^3} x_r^{p^2} z_j \in I.$$

Successively applying $\mathcal{P}^{p^2}$, $\mathcal{P}^{p^3}$, ... (resp. Sq^{2^3}, Sq^{2^4}, ...) we eventually conclude that I contains an element of the form $\sum_j c_j x_r^{p^t} z_j = x_r^{p^t}\left(\sum_j c_j z_j\right)$ with some $c_j \neq 0$, which is of the desired form. □

4. Homomorphisms

In order to show that the collection of elementary subgroup schemes detect nilpotence and projectivity, we make extensive use of homomorphisms of the form $\phi : G \to \mathbb{Z}/p$ and $\phi : G \to \mathbb{G}_{a(1)}$. Although the cohomology rings of $\mathbb{Z}/p$ and $\mathbb{G}_{a(1)}$ are identical, as in Proposition 2.1, we will continue to use different notation to distinguish the two: $H^*(\mathbb{Z}/p, k) = k[z_1] \otimes \Lambda(y_1)$ (or $k[y_1]$ if $p = 2$, with $z_1 = y_1^2$) and similarly $H^*(\mathbb{G}_{a(1)}, k) = k[x_1] \otimes \Lambda(\lambda_1)$ (or $k[\lambda_1]$ if $p = 2$, with $x_1 = \lambda_1^2$). Of particular importance will be the image $\phi^*(z_1)$ (resp. $\phi^*(x_1)$) of z_1 (resp. x_1) under the induced map in cohomology $\phi^* : H^*(\mathbb{Z}/p, k) \to H^*(G, k)$ (resp. $\phi^* : H^*(\mathbb{G}_{a(1)}, k) \to H^*(G, k)$). The significance of such homomorphisms is partially seen in the following key properties.

An essential ingredient in the proof of Theorem 6.1 is the following generalization of the Quillen-Venkov Lemma [25], whose proof is the same as that of [30, Proposition 2.3] (see also [4]). A unital rational G-algebra Λ is a unital k-algebra which admits a compatible structure of a rational G-module. That is, 1_Λ lies in Λ^G and the multiplication map $\Lambda \otimes_k \Lambda \to \Lambda$ is a map of rational G-modules. For example, take $\Lambda = \mathrm{Hom}_k(M, M)$ for a rational G-module M. For such an algebra, we denote by $\rho_\Lambda : k \to \Lambda$ the canonical homomorphism which defines 1_Λ. Further, let $\rho_\Lambda^* : H^*(G, k) \to H^*(G, \Lambda)$ denote the induced map in cohomology.

Proposition 4.1. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, Λ be an associative, unital rational G-algebra, and $\phi : G \to \mathbb{Z}/p$ (resp. $\phi : G \to \mathbb{G}_{a(1)}$) be a non-trivial homomorphism of group schemes over k. Let $z \in H^n(G, \Lambda)$ satisfy $z|_{\ker\phi} = 0$. Then z^2 is divisible by $\rho_\Lambda^*(\phi^*(z_1))$ (resp. $\rho_\Lambda^*(\phi^*(x_1))$) $\in H^2(G, \Lambda)$.*

One of the ingredients in proving Proposition 4.1 is the following well known property of the action of $H^*(H, k)$ on $H^*(H, Q)$, where H denotes either $\mathbb{Z}/p$ or $\mathbb{G}_{a(1)}$, and Q is any H-module (cf. [8], [30, 2.3]): the action of z_1 or x_1 (as appropriate) induces a periodicity isomorphism $H^j(H, Q) \xrightarrow{\sim} H^{j+2}(H, Q)$ for all $j > 0$. This fact is also the key ingredient in proving the following lemma which gives a condition for a periodicity isomorphism for $H^*(G, M)$ (for a rational G-module M) with respect

to the action of $\phi^*(z_1)$ or $\phi^*(x_1)$. This is a simple generalization of Lemma 1.2 of [8] (see also [3]) and will be used several times in the proof of Theorem 8.1.

Lemma 4.2 (Chouinard [8])**.** *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a rational G-module. Let $\phi : G \to \mathbb{Z}/p$ (resp. $\phi : G \to \mathbb{G}_{a(1)}$) be a non-trivial homomorphism of group schemes over k and let N denote the kernel of ϕ. Suppose that $H^j(N, M) = 0$ for all $j > 0$, then under the action of $H^*(G, k)$ on $H^*(G, M)$, the action of $\phi^*(z_1)$ (resp. $\phi^*(x_1)$) induces a periodicity isomorphism $H^j(G, M) \xrightarrow{\sim} H^{j+2}(G, M)$ for all $j > 0$.*

Proof. Let H denote either $\mathbb{Z}/p$ or $\mathbb{G}_{a(1)}$ and $\phi : G \twoheadrightarrow H$ be the given homorphism. Since ϕ is non-trivial we have a short exact sequence of group schemes over k:

$$1 \to N \to G \to H \to 1$$

and a corresponding Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q}(M) = H^p(H, H^q(N, M)) \Rightarrow H^{p+q}(G, M).$$

By assumption, the spectral sequence collapses to

$$E_2^{p,0}(M) = H^p(H, H^0(N, M)) \Rightarrow H^{p+q}(G, M)$$

and hence we have an isomorphism $H^*(G, M) \simeq H^*(H, Q)$ where $Q = H^0(N, M)$. The cohomology ring $H^*(H, k)$ acts on the spectral sequence with the action on the abutment via ϕ^*. Hence, the periodicity isomorphism follows from that of z_1 or x_1 on $H^*(H, Q)$. □

5. Cohomology Facts

The homomorphisms discussed in the previous section are of interest because they may be considered as elements of $H^1(G, k)$. More precisely, for any affine group scheme G/k, we have $H^1(G, k) = \mathrm{Hom}_{Gr/k}(G, \mathbb{G}_a)$ as abelian groups (cf. [30, Lemma 1.1 (a)]). This can be refined in the case that G may be identified as a semi-direct product $G = G_0 \rtimes \pi$ of an infinitesimal group scheme G_0 with a finite group π which acts on G_0 via group scheme automorphisms. In such a situation, let $\mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi$ denote those homomorphisms of group schemes over k which are preserved by the action of π. That is to say, those $\phi : G_0 \to \mathbb{G}_a$ such that $\phi(x(g)) = \phi(g)$ for all $x \in \pi$ and $g \in G_0$, where $x(g)$ denotes the image of g under the action of x on G_0.

Lemma 5.1. *Let G be an affine group scheme over k which may be identified as a semi-direct product $G = G_0 \rtimes \pi$ of an infinitesimal group scheme G_0 with a finite group π. Then $H^1(G, k) = \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi \times \mathrm{Hom}_{Gr/k}(\pi, \mathbb{G}_a)$ (as abelian groups).*

Proof. Any homomorphism $\phi : G \to \mathbb{G}_a$ determines two homomorphisms $\phi_1 : G_0 \to \mathbb{G}_a$ by $\phi_1(g) = \phi((g, 1_\pi))$ and $\phi_2 : \pi \to \mathbb{G}_a$ by $\phi_2(x) = \phi((1_{G_0}, x))$. Further, ϕ_1 necessarily preserves the action of π. Given any $g \in G_0$ and $x \in \pi$, since ϕ is a homomorphism, we have

$$\phi((x(g), x)) = \phi((x(g), 1_\pi) \cdot (1_{G_0}, x)) = \phi((x(g), 1_\pi)) + \phi((1_{G_0}, x)) = \phi_1(x(g)) + \phi_2(x).$$

On the other hand, we also have

$$\phi((x(g), x)) = \phi((1_{G_0}, x) \cdot (g, 1_\pi)) = \phi((1_{G_0}, x)) + \phi((g, 1_\pi)) = \phi_2(x) + \phi_1(g).$$

Hence, we must have $\phi_1(x(g)) = \phi_1(g)$ as claimed.

Conversely, given homomorphisms $\phi_1 \in \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi$ and $\phi_2 \in \mathrm{Hom}_{Gr/k}(\pi, \mathbb{G}_a)$, define a map $\phi : G \to \mathbb{G}_a$ by $\phi((g, x)) = \phi_1(g) + \phi_2(x)$. For any pair $(g_1, x_1), (g_2, x_2) \in G$, we check that ϕ is in fact a homomorphism. On the one hand, we have

$$\begin{aligned}\phi((g_1, x_1) \cdot (g_2, x_2)) &= \phi((g_1 \cdot x_1(g_2), x_1 \cdot x_2) = \phi_1(g_1 \cdot x_1(g_2)) + \phi_2(x_1 \cdot x_2)\\ &= \phi_1(g_1) + \phi_1(x_1(g_2)) + \phi_2(x_1) + \phi_2(x_2).\end{aligned}$$

On the other hand,

$$\phi((g_1, x_1)) + \phi((g_2, x_2)) = \phi_1(g_1) + \phi_2(x_1) + \phi_1(g_2) + \phi_2(x_2).$$

Since $\phi_1(x_1(g_2)) = \phi_1(g_2)$, these agree and ϕ is a homomorphism. □

The first step in the proofs of the main results is to reduce to the case that G is a unipotent group scheme. This allows us to make use of the following result of [30] which is a generalization of a classical result for finite p-groups (cf. [13, Theorem 7.2.4]).

Lemma 5.2 ([30, Lemma 1.2]). *Let $\psi : G \to H$ be a surjective homomorphism of unipotent group schemes over k. If $\psi^* : H^1(H, k) \to H^1(G, k)$ is an isomorphism and if $\psi^* : H^2(H, k) \to H^2(G, k)$ is injective, then ψ is an isomorphism.*

For unipotent group schemes there is a nice cohomological criterion (Proposition 5.3) for detecting projectivity which is well known for p-groups and holds more generally. We remind the reader that for a finite group G a kG-module is projective if and only if it is injective. More generally, any finite dimensional cocommutative Hopf algebra is a Frobenius algebra (cf. [19], [21]) and hence a module over such an algebra is in fact projective if and only if it is injective (cf. [14]). So this equivalence holds for arbitrary finite group schemes. Further, the theory of finite dimensional algebras shows that if k is the only simple module for such an algebra (e.g., for a unipotent group scheme), then the algebra is indecomposable as a module over itself, and hence a module is in fact projective if and only if it is free. Thus, over a finite unipotent group scheme, the notions of projective, injective, and free are all equivalent.

Proposition 5.3 (cf. [4] or [3]). *Let k be a field of characteristic $p > 0$ and G be a finite unipotent group scheme over k. For any rational G-module M, M is projective (= injective = free) if and only if $H^1(G, M) = 0$.*

In order to reduce to the case of a unipotent group scheme, we use the injectivity of certain restriction maps in cohomology. If π is a finite group and $\pi_S \subset \pi$ is a p-Sylow subgroup, then for any $k\pi$-module M the restriction map in cohomology $H^*(\pi, M) \to H^*(\pi_S, M)$ is an injection (cf. [4] or [13]). This can be partially extended to the more general setting of finite group schemes.

Lemma 5.4. *Let k be a field of characteristic $p > 0$ and G be a finite group scheme over k which can be identified as a semi-direct product $G = G_0 \rtimes \pi$ of an infinitesimal group scheme G_0 (which is a closed subgroup scheme in G) and a finite group π. Further, let $\pi_S \subset \pi$ be a p-Sylow subgroup of π and M be a rational G-module. Then the restriction map $H^*(G, M) \to H^*(G_S, M)$ for the embedding $G_S = G_0 \rtimes \pi_S \hookrightarrow G_0 \rtimes \pi = G$ is an injection.*

Proof. The idea is to identify G-cohomology groups with certain π-cohomology groups. We first note that there exists a projective resolution $X_\bullet \twoheadrightarrow k$ of rational G_0-modules which admits a compatible action of the finite group π. That is, this is also a complex of π-modules and for any $x \in \pi$, $g \in G_0$, and $m \in X_n$ (any n) we have $x \cdot (g \cdot m) = x(g) \cdot (x \cdot m)$. Consider the cobar resolution (cf. [19])

$$k \to k[G] \to k[G]^{\otimes 2} \to \dots$$

which is an injective resolution of k over G. Further, this is a sequence of rational G-modules and maps if G is defined to act by the left regular representation on the first $k[G]$ factor of each term $k[G]^{\otimes n}$ (cf. [19]). Hence $\pi \subset G$ acts on the sequence and does so compatibly with respect to its action on G_0. Let $X_\bullet$ be the dual complex (i.e., take the k-linear dual of each module and reverse the arrows). Then $X_\bullet$ is necessarily a projective resolution of k over G, which can be considered as a *resolution* over G_0 on which π acts compatibly. Finally, since G_0 is a closed subgroup scheme of G, any projective G-module is also a projective G_0-module (cf. [19]). So $X_\bullet$ is in fact a *projective* resolution of k over G_0 with π acting compatibly.

Given that such a resolution exists, the same argument as on p. 19 of [13] (for a semi-direct product of finite groups) shows that there is an isomorphism

$$H^*(G, M) = H^*(G_0 \rtimes \pi, M) \simeq H^*(\pi, \mathrm{Hom}_{G_0}(X_\bullet, M))$$

where the latter cohomology group is a *hypercohomology* group. That is, the coefficients consist of a cochain complex of modules. Further, this isomorphism is preserved under the natural embedding $\pi_S \hookrightarrow \pi$ so that the following diagram commutes:

$$\begin{tikzcd}
H^*(G_0 \rtimes \pi, M) \arrow[r, "\sim"] \arrow[d, "res"'] & H^*(\pi, \mathrm{Hom}_{G_0}(X_\bullet, M)) \arrow[d, "res"] \\
H^*(G_0 \rtimes \pi_S, M) \arrow[r, "\sim"] & H^*(\pi_S, \mathrm{Hom}_{G_0}(X_\bullet, M)).
\end{tikzcd}$$

Finally, the left hand map will be injective if the right hand map is. However, the injectivity of the right hand restriction map in ordinary cohomology can be extended to hypercohomology. For example, one standard proof in ordinary cohomology (cf. [4] or [13]) is based on the fact that for a $k\pi$-module N, the composite

$$H^*(\pi, N) \xrightarrow{res} H^*(\pi_S, N) \xrightarrow{Tr} H^*(\pi, N)$$

is multiplication by the index $[\pi : \pi_S]$ which is invertible in k, where Tr denotes the transfer map. Hence the composite is an injection and so the restriction map is also. This same argument works with N replaced by a cochain complex.

Alternately, by the Eckmann-Shapiro lemma, we may identify

$$H^*(\pi_S, N) = \mathrm{Ext}^*_{\pi_S}(k, N) \simeq \mathrm{Ext}^*_{\pi}(k\pi \otimes_{k\pi_S} k, N),$$

and the restriction map

$$H^*(\pi, N) = \mathrm{Ext}^*_{\pi}(k, N) \xrightarrow{res} \mathrm{Ext}^*_{\pi_S}(k, N) \simeq \mathrm{Ext}^*_{\pi}(k\pi \otimes_{k\pi_S} k, N)$$

is simply the map induced from the canonical map $k\pi \otimes_{k\pi_S} k \to k$ which sends $x \otimes c \mapsto c$. This module map is split by the map $k \to k\pi \otimes_{k\pi_S} k$ which sends $1 \mapsto \frac{1}{|T|} \sum_{t \in T} t \otimes 1$, where $T \subset \pi$ is a set of left coset representatives of π_S in π. This

splitting of the module map induces a splitting of the restriction map in cohomology when N is either a module or a complex. □

6. The Restriction Theorem in Cohomology

In this section we present a generalization of Theorem 1.4 on detecting nilpotent cohomology classes, which is also an extension of Theorem 2.5 of [30] which applies to infinitesimal unipotent group schemes.

Theorem 6.1. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and Λ be an associative, unital rational G-algebra (as defined in Section 4). Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent. If $z \in H^n(G, \Lambda)$ satisfies the property that for any field extension K/k and any group scheme embedding $\nu : \mathcal{E}_{r,s} \otimes_k K \hookrightarrow G \otimes_k K$ over K the cohomology class $\nu^*(z) \in H^n(\mathcal{E}_{r,s} \otimes_k K, \Lambda \otimes_k K)$ is nilpotent, then z is itself nilpotent.*

Proof. The strategy of the proof is to extend that of Theorem 2.5 in [30]. For any field extension K/k, since $H^*(G \otimes_k K, \Lambda \otimes_k K) = H^*(G, \Lambda) \otimes_k K$ (cf. [19]), if z is nilpotent after base change, then it is necessarily nilpotent. Hence it suffices to assume that k is algebraically closed. In this case, the points of G are certainly k-rational and so, as noted in the Introduction, we can identify G as the semi-direct product $G_0 \rtimes \pi$ where $\pi = G(k)$ is the finite group of k-points of G. By assumption, G_0 is unipotent, but π may be an arbitrary finite group. However, by Lemma 5.4, we may assume that π is in fact a p-group, for, if z is nilpotent after restriction to the subgroup $G_0 \rtimes \pi_S$ (for a p-Sylow subgroup π_S of π), since the restriction map in cohomology is injective, z must necessarily be nilpotent. Hence, the group G may be assumed to be unipotent.

We now proceed by induction on $\dim_k k[G]$ and are trivially done if $\dim_k k[G] = 1$ or $G = \mathcal{E}_{r,s}$, and so assume that the theorem holds for all groups H over any field K/k with $\dim_K K[H] < \dim_k k[G]$. As in Section 4, let $\rho_\Lambda^* : H^*(G, k) \to H^*(G, \Lambda)$ denote the map induced by the module map $\rho_\Lambda : k \to \Lambda$ which defines 1_Λ.

The next step is to reduce to the case that π is elementary abelian. If not, by Serre's theorem (Theorem 3.1), there exists a finite product $u = \prod u_i = \prod \beta(v_i) \in H^*(\pi, k)$ which is zero in $H^*(\pi, k)$ for non-zero $v_i \in H^1(\pi, k)$. Each v_i can be considered as a non-zero map $\pi \to \mathbb{Z}/p$, which can be extended to $\phi_i : G \twoheadrightarrow \pi \twoheadrightarrow \mathbb{Z}/p$. Let N_i denote the kernel of ϕ_i. Since ϕ_i is non-trivial, $\dim_k k[N_i] < \dim_k k[G]$ and so by induction, the restriction of z to each N_i is nilpotent. Hence, by Proposition 4.1, z^2 is divisible by $\rho_\Lambda^*(\phi_i^*(z_1))$ for each i. Consider the canonical projection $\psi : G \to \pi$ and the induced map $\psi^* : H^*(\pi, k) \to H^*(G, k)$. Since $\psi^*(u_i) = \phi_i^*(z_1)$ for each i, some power of z is divisible by $\rho_\Lambda^*(\psi^*(u))$ and hence is zero.

Hence, we may assume that $\pi = E_s$ for some s (possibly zero). In the trivial case $s = 0$, we have $G = G_0$ and the claim is precisely Theorem 2.5 of [30]. In any case, the succeeding argument holds in general (with Case I being impossible in that situation). The remainder of the argument consists of three cases depending on $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi$ and uses arguments similar to those in the above reduction to the case that π is elementary abelian.

CASE I: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi = 0$

Consider the natural projection $\psi : G = G_0 \rtimes \pi \to \pi$ and the induced map $\psi^* : H^*(\pi, k) \to H^*(G, k)$. By the assumption and Lemma 5.1, we have an isomorphism

$\psi^* : H^1(\pi, k) \to H^1(G, k)$. Consider the map in degree 2, $\psi^* : H^2(\pi, k) \to H^2(G, k)$. By Lemma 5.2, either $G \simeq \pi \simeq E_s$ and we are done or $I \equiv \ker(\psi^*)$ contains some non-zero element of degree 2. Now, the map ψ^* preserves the action of the Steenrod algebra (cf. [12], [27], and also [30]) and hence I is a homogeneous ideal stable under the action of the Steenrod operations. By Proposition 3.4, there exists a non-zero product $u = \prod u_i = \prod \beta(v_i) \in H^*(\pi, k)$ which lies in I. In other words, the image of u under ψ^* is zero in $H^*(G, k)$. On the other hand, arguing as above, by Proposition 4.1, we conclude that some power of z is divisible by $\rho_\Lambda^*(\psi^*(u))$ and hence is zero.

CASE II: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi = 1$

Let $\theta : G_0 \to \mathbb{G}_a$ be a representative map preserved by $\pi = E_s$. Since G_0 is infinitesimal, the image of θ lands in some $\mathbb{G}_{a(r)}$. If $r > 1$, the composition

$$G_0 \xrightarrow{\theta} \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a,$$

where F is the Frobenius morphism, would give another linearly independent homomorphism preserved by π. Hence we must have $\theta : G_0 \to \mathbb{G}_{a(1)}$. Consider the homomorphism $\eta : G = G_0 \rtimes E_s \xrightarrow{\theta \times Id} \mathbb{G}_{a(1)} \times E_s = \mathcal{E}_{1,s}$ and the induced morphism $\eta^* : H^*(\mathcal{E}_{1,s}, k) \to H^*(G, k)$. Specifically, consider the map on H^1 (cf. Lemma 5.1):

$$\begin{aligned} H^1(\mathcal{E}_{1,s}, k) = &\mathrm{Hom}_{Gr/k}(\mathbb{G}_{a(1)}, \mathbb{G}_a) \times \mathrm{Hom}_{Gr/k}(E_s, \mathbb{G}_a) \\ &\to \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^{E_s} \times \mathrm{Hom}_{Gr/k}(E_s, \mathbb{G}_a) = H^1(G, k). \end{aligned}$$

Again, this is evidently an isomorphism and as before, either $G \simeq \mathcal{E}_{1,s}$ and we're done or there exists a non-zero element of degree 2 in $I = \ker(\eta^*)$. In the latter case, by Proposition 3.6, there exists a product $u = x_1^n \prod u_i \in H^*(\mathcal{E}_{1,s}, k)$ which lies in I.

Each element u_i may again be identified with $\beta(v_i)$ for some non-zero $v_i \in H^1(E_s, k)$. Just as above, for each i, we consider the homomorphism $\phi_i : G \twoheadrightarrow E_s \twoheadrightarrow \mathbb{Z}/p$ corresponding to v_i and conclude by Proposition 4.1 that z^2 is divisible by $\rho_\Lambda^*(\eta^*(u_i))$.

On the other hand, consider the composite

$$\phi : G = G_0 \rtimes E_s \xrightarrow{\eta} \mathbb{G}_{a(1)} \times E_s \twoheadrightarrow \mathbb{G}_{a(1)},$$

where the last map is the canonical projection. By Proposition 4.1, z^2 is also divisible by $\rho_\Lambda^*(\eta^*(x_1))$. Hence some power of z is divisible by $\rho_\Lambda(\eta^*(u))$ and thus zero since $\eta^*(u)$ is zero in $H^*(G, k)$.

CASE III: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi > 1$

This case must be further divided into two, based on $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^\pi$. As noted above, since G_0 is infinitesimal, any $\phi : G_0 \to \mathbb{G}_a$ has image in some $\mathbb{G}_{a(r)}$. So there is certainly one non-trivial π-preserved map $G_0 \to \mathbb{G}_{a(1)}$, but there need not be two linearly independent such maps.

CASE III(a): $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^\pi = 1$

The proof of this case is similar to the proof of Theorem 1.6 in [30]. Let $\theta : G_0 \to \mathbb{G}_{a(1)}$ be a non-trivial representative map preserved by π. Since θ is preserved by the action of π, it can be extended to a non-trivial map $\phi : G \to \mathbb{G}_{a(1)}$ by $\phi((g, x)) = \theta(g)$. The Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$ induces via composition a map (of the same name) $F : \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a) \to \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)$. Further, since any π-preserved map remains so after composition with F, this restricts to a map $F : \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi \to \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi$, and we may identify the kernel of this

(restricted) map with $\mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^{\pi}$ (cf. [30, Lemma 1.1]). By assumption, the kernel must then be one-dimensional, and so there exists a non-negative integer r and homomorphism $\zeta \in \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(r)})^{\pi}$ with $F^{r-1}(\zeta) = \theta$ and such that the set $\{\zeta, F(\zeta), \ldots, F^{r-1}(\zeta)\}$ is a basis for the subspace $\mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^{\pi} \subset H^1(G, k)$.

Consider the homomorphism $\eta : G = G_0 \rtimes E_s \xrightarrow{\zeta \times Id} \mathbb{G}_{a(r)} \times E_s$ and the induced map on cohomology $\eta^* : H^*(\mathcal{E}_{r,s}, k) \to H^*(G, k)$. On H^1 the map

$$\begin{aligned} H^1(\mathcal{E}_{r,s}, k) = &\mathrm{Hom}_{Gr/k}(\mathbb{G}_{a(r)}, \mathbb{G}_a) \times \mathrm{Hom}_{Gr/k}(E_s, \mathbb{G}_a) \\ &\to \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^{E_s} \times \mathrm{Hom}_{Gr/k}(E_s, \mathbb{G}_a) = H^1(G, k) \end{aligned}$$

is the identity on the right factor and on the left factor maps the basis $\lambda_1, \lambda_2, \ldots, \lambda_r$ to the basis $\zeta, F(\zeta), \ldots, F^{r-1}(\zeta) = \theta$. Hence, this is an isomorphism and either $G \simeq \mathcal{E}_{r,s}$ and we're done or there is a homomorphism $\eta : G \to \mathcal{E}_{r,s}$ for which (by Proposition 3.6) the kernel of $\eta^* : H^*(\mathcal{E}_{r,s}, k) \to H^*(G, k)$ contains an element of the form $u = x_r^n \prod u_i$ with $\eta^*(x_r) = \theta^*(x_1) = \phi^*(x_1)$ and for each i, $\eta^*(u_i) = \phi_i^*(z_1)$ for some non-trivial $\phi_i : G \to \mathbb{Z}/p$. By Proposition 4.1 and the usual argument, some power of z is divisible by $\rho_\Lambda^*(\eta^*(u))$ and hence is zero.

CASE III(b): $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^{\pi} > 1$

Let $\theta_1 : G_0 \to \mathbb{G}_{a(1)}$ and $\theta_2 : G_0 \to \mathbb{G}_{a(1)}$ be two linearly independent (and non-trivial) homomorphisms which are preserved by the action of π. Since they are preserved by π, these maps can be extended to maps $\phi_1 : G \to \mathbb{G}_{a(1)}$ by $\phi_1((g, x)) = \theta_1(g)$ and $\phi_2 : G \to \mathbb{G}_{a(1)}$ by $\phi_2((g, x)) = \theta_2(g)$. Clearly these remain non-trivial and linearly independent maps $G \to \mathbb{G}_{a(1)}$. Now, the identical argument as in the case $\dim_k \mathrm{Hom}_{Gr/k}(G, \mathbb{G}_{a(1)}) > 1$ in the proof of [30, Theorem 2.5] may be applied with ϕ_1 and ϕ_2 to imply that z is indeed nilpotent. $\square$

Question 6.2. Can this result be extended to any finite group scheme G?

Remark 6.3. Given a homomorphism $\mathcal{E}_{r,s} \to G$ of group schemes over k, the image of $\mathcal{E}_{r,s}$ is necessarily a closed subgroup and moreover isomorphic to an elementary group scheme $\mathcal{E}_{r',s'}$ for some $r' \leq r, s' \leq s$. Hence, Theorem 6.1 (as well as succeeding results) could be stated either in terms of homomorphisms of the form $\mathcal{E}_{r,s} \to G$ or in terms of closed subgroup schemes of the form $\mathcal{E}_{r,s}$.

7. Consequences

In this section, we note some immediate consequences of Theorem 6.1. First, we note a slightly weaker version of the theorem, which is stated in terms of Ext-groups like Theorem 1.4. Given a rational G-module M, the algebra $\Lambda \equiv \mathrm{Hom}_k(M, M)$ is an associative, unital rational G-algebra, and hence the theorem applies to Λ. Further, there is a natural isomorphism $H^*(G, \Lambda) \simeq \mathrm{Ext}_G^*(M, M)$. In the case that M is finite dimensional, for any field extension K/k, we have $\mathrm{Ext}^*_{G \otimes_k K}(M \otimes_k K, M \otimes_k K) \simeq \mathrm{Ext}_G^*(M, M) \otimes_k K$. Hence, the following is an immediate consequence of Theorem 6.1.

Corollary 7.1. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a finite dimensional rational G-module. Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent. If $z \in$*

$\mathrm{Ext}^n_G(M,M)$ *satisfies the property that for any field extension K/k and any group scheme embedding $\nu : \mathcal{E}_{r,s} \otimes_k K \hookrightarrow G \otimes_k K$ over K the cohomology class $\nu^*(z) \in \mathrm{Ext}^n_{\mathcal{E}_{r,s}\otimes_k K}(M \otimes_k K, M \otimes_k K)$ is nilpotent, then z is itself nilpotent.*

A standard argument (cf. [4], [30]) which is sketched below shows that a generalization of Chouinard's theorem (Theorem 1.3) follows from Corollary 7.1.

Corollary 7.2. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a finite dimensional rational G-module. Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent. Then M is projective as a G-module if and only if for every field extension K/k and closed subgroup scheme $H \subset G \otimes_k K$ with $H \simeq \mathcal{E}_{r,s} \otimes_k K$ the restriction of $M \otimes_k K$ to H is projective.*

Proof. If M is projective over G, then it remains so upon restriction to any closed subgroup scheme (cf. [19]). Conversely, given an $H \subset G \otimes_k K$ with $H \simeq \mathcal{E}_{r,s} \otimes_k K$, if $M \otimes_k K$ is projective upon restriction, then $\mathrm{Ext}^i_H(M \otimes_k K, M \otimes_k K) = 0$ for all $i > 0$. Hence, by Corollary 7.1, every element $z \in \mathrm{Ext}^n_G(M,M)$ for $n > 0$ is nilpotent. Since, by [17], $\mathrm{Ext}^*_G(M,M)$ is finitely generated over the Noetherian ring $H^{2*}(G,k) = \mathrm{Ext}^{2*}_G(k,k)$, we must have $\mathrm{Ext}^i_G(M,M) = 0$ for all $i > N$ for some N. As the notions of projective and injective are equivalent (see the discussion preceding Proposition 5.3), it follows that M must in fact be projective. $\square$

Let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over k with restricted enveloping algebra $u(\mathfrak{g})$. As mentioned in the Introduction, this corresponds to a certain (height 1) infinitesimal group scheme. Theorem 6.1 applied to the corresponding group scheme says that nilpotence is detected by the collection of subalgebras $u(\langle x\rangle) \subset u(\mathfrak{g} \otimes_k K)$ for each p-nilpotent element $x \in \mathfrak{g} \otimes_k K$. Here $u(\langle x\rangle)$ denotes the subalgebra generated by x and is simply isomorphic to $K[x]/(x^p)$ as an algebra. Using this special case, we can obtain a restriction theorem in terms of subalgebras of the form $K[x]/(x^p)$.

Corollary 7.3. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a finite dimensional rational G-module. Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent. If $z \in \mathrm{Ext}^n_G(M,M)$ satisfies the property that for any field extension K/k and any embedding of algebras $\nu : B \hookrightarrow K[G \otimes_k K]^*$ with $B \simeq K[x]/(x^p)$ the cohomology class $\nu^*(z) \in \mathrm{Ext}^n_B(M \otimes_k K, M \otimes_k K)$ is nilpotent, then z is itself nilpotent.*

Proof. For any closed subgroup scheme of the form $\mathcal{E}_{r,s} \otimes_k K \subset G \otimes_k K$, the corresponding Hopf algebra $A = K[\mathcal{E}_{r,s} \otimes_k K]^*$ is isomorphic as an algebra to the restricted enveloping algebra, $u(\mathfrak{g})$, of an abelian Lie algebra $\mathfrak{g}$ over K with trivial p-mapping. Given an A-module N, let $\tilde{N}$ denote the module N considered as a $u(\mathfrak{g})$-module. As Yoneda algebras, we necessarily have $\mathrm{Ext}^*_A(N,N) \simeq \mathrm{Ext}^*_{u(\mathfrak{g})}(\tilde{N},\tilde{N})$. Hence, an element of $\mathrm{Ext}^*_A(N,N)$ is nilpotent if and only if it is nilpotent when considered as an element of $\mathrm{Ext}^*_{u(\mathfrak{g})}(\tilde{N},\tilde{N})$. As nilpotence is detected by certain Hopf subalgebras of the form $K[x]/(x^p)$ in the latter case, this is also true in the former. So, if z is nilpotent upon restriction to all subalgebras $K[x]/(x^p)$, it is nilpotent upon restriction to each $\mathcal{E}_{r,s} \otimes_k K$ and hence, by Corollary 7.1, is itself nilpotent. $\square$

Remark 7.4. Evidently it suffices to take only some of the subalgebras of the form $K[x]/(x^p)$, i.e., those which are "contained in" an elementary subgroup scheme of G.

8. Detecting Projectivity

As seen in the previous section, an almost immediate corollary of the restriction theorem on nilpotent cohomology classes (Theorem 6.1) is a generalization of Chouinard's theorem (Theorem 1.3) for finite dimensional modules (Corollary 7.2). In this section, we show that essentially the same proof as for Theorem 6.1 can be used to extend this result to infinite dimensional modules, by replacing applications of Proposition 4.1 with applications of Lemma 4.2.

Theorem 8.1. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a rational G-module. Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent and that all the points of G are k-rational. Then M is projective as a G-module if and only if for every field extension K/k and closed subgroup scheme $H \subset G \otimes_k K$ with $H \simeq \mathcal{E}_{r,s} \otimes_k K$ the restriction of $M \otimes_k K$ to H is projective.*

Remark 8.2. If M is assumed to be finite dimensional, then the proof shows that it suffices to take the single field extension $K = \bar{k}$, the algebraic closure of k. In this sense, it is a slightly stronger result than Corollary 7.2 and more comparable to Proposition 7.6 of [30] (as well as to Chouinard's theorem).

Proof. As previously noted, if M is projective over G, then it remains so upon restriction to any closed subgroup scheme (cf. [19]). Conversely, suppose that all restrictions are projective. The outline of the proof is the same as for the proof of Theorem 6.1 with the details modified along the lines of the proof of the Theorem in [3] which was based on the original arguments of Chouinard [8].

By the assumption on the points of G (see the Introduction), we may write $G = G_0 \rtimes \pi$ as usual, with G_0 assumed to be unipotent. The first step is to reduce to the case that π is a p-group. Let $\pi_S \subset \pi$ be a p-Sylow subgroup of π, and let G_S denote the subgroup $G_0 \rtimes \pi_S \subset G$. To show that M is projective, it suffices to show that $\mathrm{Ext}^i_G(M, N) \simeq H^i(G, \mathrm{Hom}_k(M, N)) = 0$ for all $i > 0$ and any rational G-module N. If M is projective over G_S, then $H^i(G_S, \mathrm{Hom}_k(M, N)) = 0$ for all $i > 0$, and since, by Lemma 5.4, the restriction map $H^*(G, \mathrm{Hom}_k(M, N)) \to H^*(G_S, \mathrm{Hom}_k(M, N))$ is an injection, we also have $H^i(G, \mathrm{Hom}_k(M, N)) = 0$ for all $i > 0$. Thus it suffices to assume that π is a p-group and G is in fact unipotent.

Under the assumption that G is unipotent, to show that M is projective over G, it suffices, by Proposition 5.3, to show that $H^1(G, M) = 0$. If L/k is any field extension, then $H^*(G \otimes_k L, M \otimes_k L) = H^*(G, M) \otimes_k L$ (cf. [19]). Hence, if it can be shown that $H^1(G \otimes_k L, M \otimes_k L) = 0$ for some field extension L/k, then we also have $H^1(G, M) = 0$. So it suffices to assume that k is algebraically closed.

We now proceed by induction on $\dim_k k[G]$ and are trivially done if $\dim_k k[G] = 1$ or $G = \mathcal{E}_{r,s}$, and so assume that the theorem holds for all groups H over any field K/k with $\dim_K K[H] < \dim_k k[G]$.

As in the proof of Theorem 6.1, the next step is to reduce the problem to the case that π is elementary abelian. If π is not elementary abelian, by Serre's theorem

(Theorem 3.1), there exists a finite product $u = \prod u_i = \prod \beta(v_i) \in H^*(\pi, k)$ which is zero in $H^*(\pi, k)$ for non-zero $v_i \in H^1(\pi, k)$. Each v_i can be considered as a non-zero map $\pi \twoheadrightarrow \mathbb{Z}/p$, which can be extended to $\phi_i : G \twoheadrightarrow \pi \twoheadrightarrow \mathbb{Z}/p$. Let N_i denote the kernel of ϕ_i. Since ϕ_i is non-trivial, $\dim_k k[N_i] < \dim_k k[G]$ and so by induction M is projective upon restriction to N_i for each i. Hence, $H^j(N_i, M) = 0$ for all $j > 0$ and each i, and so by Lemma 4.2 the action of $\phi_i^*(z_1) = \psi^*(u_i)$ (where $\psi^* : H^*(\pi, k) \to H^*(G, k)$ is induced from the canonical projection $\psi : G \to \pi$) induces a periodicity isomorphism $H^j(G, M) \to H^{j+2}(G, M)$ for all $j > 0$. Hence, the product $\prod_i \phi_i^*(u_i) = \psi^*(u) \in H^*(G, k)$ acts via an isomorphism on $H^1(G, M)$. But, $\psi^*(u)$ is zero and so we must have $H^1(G, M) = 0$. Thus M is projective.

From now on we may suppose that $\pi = E_s$ for some s (possibly zero). In the trivial case $s = 0$, we have $G = G_0$ and the claim is precisely the Theorem in [3]. In any case, the following argument still holds (with Case I being impossible in that situation). As in 6.1, the rest of the proof consists of three steps based on $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi$ with arguments like the preceding one.

CASE I: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi = 0$

Consider the natural projection $\psi : G = G_0 \rtimes \pi \to \pi$ and the induced map $\psi^* : H^*(\pi, k) \to H^*(G, k)$. By assumption and Lemma 5.1, we have an isomorphism $\psi^* : H^1(\pi, k) \to H^1(G, k)$. Consider the map in degree 2, $\psi^* : H^2(\pi, k) \to H^2(G, k)$. By Lemma 5.2, either $G \simeq \pi \simeq E_s$ and we are done or $I \equiv \ker(\psi^*)$ contains some non-zero element of degree 2. Again, the map ψ^* preserves the action of the Steenrod algebra and hence I is a homogeneous ideal stable under the action of the Steenrod operations. By Proposition 3.4, there exists a non-zero product $u = \prod u_i = \prod \beta(v_i) \in H^*(\pi, k)$ which lies in I. In other words, its image under ψ^* is zero in $H^*(G, k)$. On the other hand, applying the above argument to u, we conclude that $\psi^*(u)$ acts isomorphically on $H^1(G, M)$ and hence $H^1(G, M) = 0$.

CASE II: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi = 1$

Just as in 6.1, there is a homomorphism $\eta : G \to \mathbb{G}_{a(1)} \times E_s = \mathcal{E}_{1,s}$ which is either an isomorphism (in which case we're done) or there exists a non-zero element of degree 2 in $I = \ker\{\eta^* : H^*(\mathcal{E}_{1,s}, k) \to H^*(G, k)$. In the latter case, by Proposition 3.6, there exists a product $u = x_1^n \prod u_i \in H^*(\mathcal{E}_{1,s}, k)$ which lies in I.

Each element u_i may again be identified with $\beta(v_i)$ for some non-zero $v_i \in H^1(E_s, k)$. Just as above, for each i, we consider the homomorphism $\phi_i : G \twoheadrightarrow E_s \twoheadrightarrow \mathbb{Z}/p$ corresponding to v_i and conclude that the action of $\eta^*(u_i) = \phi_i^*(z_1) \in H^2(G, k)$ on $H^*(G, M)$ gives an isomorphism $H^j(G, M) \xrightarrow{\sim} H^{j+2}(G, M)$ for all $j > 0$.

On the other hand, consider the composite

$$\phi : G = G_0 \rtimes E_s \xrightarrow{\eta} \mathbb{G}_{a(1)} \times E_s \twoheadrightarrow \mathbb{G}_{a(1)},$$

where the last map is the canonical projection. By Lemma 4.2, $\phi^*(x_1) = \eta^*(x_1)$ induces a periodicity isomorphism $H^j(G, M) \xrightarrow{\sim} H^{j+2}(G, M)$ for all $j > 0$. Hence $\eta^*(u)$ acts isomorphically on $H^1(G, M)$. Since $\eta^*(u)$ is zero in $H^*(G, k)$ we must have that $H^1(G, M) = 0$.

CASE III: $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_a)^\pi > 1$

This case is again divided into two sub-cases based on $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^\pi$.

CASE III(a): $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^\pi = 1$

As in 6.1, we deduce that there is a homomorphism $\eta : G \to \mathbb{G}_{a(r)} \times E_s = \mathcal{E}_{r,s}$ which is either an isomorphism (in which case we're done) or the kernel of $\eta^* : H^*(\mathcal{E}_{r,s}, k) \to H^*(G, k)$ contains a non-zero element of degree 2. In the latter case, by Proposition 3.6, the kernel contains an element of the form $u = x_r^n \prod u_i$ with $\eta^*(x_r) = \phi^*(x_1)$ for some non-trivial $\phi : G \to \mathbb{G}_{a(1)}$ and $\eta^*(u_i) = \phi_i^*(z_1)$ for some non-trivial $\phi_i : G \to \mathbb{Z}/p$ for each i. As above, the actions of $\eta^*(x_r)$ and $\eta^*(u_i)$ (for each i) induce a periodicity isomorphism $H^j(G, M) \to H^{j+2}(G, M)$ for all $j > 0$. Hence, $\eta^*(u)$ acts isomorphically, but since it's zero we must have $H^1(G, M) = 0$.

CASE III(b): $\dim_k \mathrm{Hom}_{Gr/k}(G_0, \mathbb{G}_{a(1)})^\pi > 1$

Let θ_1 and θ_2 be two linearly independent and non-trivial homomorphisms $G_0 \to \mathbb{G}_{a(1)}$ which preserve the action of π. These can then be extended to two linearly independent and non-trivial homomorphisms $\phi_1, \phi_2 : G \to \mathbb{G}_{a(1)}$. We now apply the same argument as in [3] to the maps ϕ_1 and ϕ_2.

For any homorphism $\phi : G \to \mathbb{G}_{a(1)}$, let x_ϕ denote the image of the canonical generator $x_1 \in H^2(\mathbb{G}_{a(1)}, k)$ under the induced map in cohomology. Once again, the inductive argument as above shows that both $x_{\phi_1} = \phi_1^*(x_1)$ and $x_{\phi_2} = \phi_2^*(x_1)$ induce periodicity isomorphisms $H^j(G, M) \xrightarrow{\sim} H^{j+2}(G, M)$ for all $j > 0$. Moreover, for any $c_1, c_2 \in k$, by Corollary 1.5 of [30],

$$c_1 x_{\phi_1} + c_2 x_{\phi_2} = x_{c_1^{1/p}\phi_1} + x_{c_2^{1/p}\phi_2} = x_{c_1^{1/p}\phi_1 + c_2^{1/p}\phi_2} \in H^2(G, k).$$

If at least one of c_1, c_2 is non-zero, by linear independence, the map $c_1^{1/p}\phi_1 + c_2^{1/p}\phi_2 : G \to \mathbb{G}_{a(1)}$ is non-trivial and hence by Lemma 4.2, the element $c_1 x_{\phi_1} + c_2 x_{\phi_2}$ also induces a periodicity isomorphism $H^1(G, M) \xrightarrow{\sim} H^3(G, M)$. On the contrary, since k is by assumption algebraically closed, if M is finite-dimensional, an eigenvalue argument implies that for some c_1, c_2 with not both zero, this map will not be an isomorphism unless $H^1(G, M) = 0$. Thus, for finite dimensional modules the proof is complete and we see that it is only necessary to extend to the algebraic closure of k.

For infinite dimensional M, this eigenvalue argument need not work and must be replaced by an infinite dimensional substitute used in [6] and which requires a further field extension. Let K/k be any non-trivial algebraically closed field extension. After base change, the (extended) maps $\phi_1, \phi_2 : G \otimes_k K \to \mathbb{G}_{a(1)} \otimes_k K$ remain linearly independent. Hence, as the inductive arguments apply equally well over K, we again conclude that for any $c_1, c_2 \in K$ with not both zero, the element

$$c_1 x_{\phi_1} + c_2 x_{\phi_2} = x_{c_1^{1/p}\phi_1 + c_2^{1/p}\phi_2} \in H^2(G \otimes_k K, K)$$

also induces a periodicity isomorphism

$$H^1(G \otimes_k K, M \otimes_k K) = H^1(G, M) \otimes_k K \to H^3(G \otimes_k K, M \otimes_k K) = H^3(G, M) \otimes_k K.$$

Applying Lemma 4.1 of [6] (or Lemma 2 of [3]) we conclude that $H^1(G, M) = 0$ as desired. $\square$

Question 8.3. Can this result be extended to any finite group scheme G?

We end this section by recalling Dade's Lemma [11, Lemma 11.8] which says that projectivity (or freeness) of a finite dimensional module for an elementary abelian group E can be detected on "cyclic shifted subgroups", that is, on certain subalgebras

of kE of the form $k[x]/(x^p)$. Indeed, this is simply a statement about detecting projectivity (or equivalently freeness) of modules over a truncated polynomial algebra $k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$. However, Dade's lemma as originally stated holds only for finite dimensional modules. In [6], D. Benson, J. Carlson, and J. Rickard extend Dade's Lemma to arbitrary modules but at the cost of having to consider field extensions. Indeed, they note that the lemma fails in general without this assumption. We restate it here from a purely algebra perspective.

Proposition 8.4 (Dade's Lemma – [6, Theorem 5.2])**.** *Let k be a field of characteristic $p > 0$, $n > 0$ be an integer, $A_n = k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$ be a truncated polynomial algebra over k, and M be an A_n-module. Then M is projective over A_n if and only if for every field extension K/k, the restriction of $M \otimes_k K$ to the subalgebra $K[x]/(x^p) \subset A_n \otimes_k K$ is free for each non-zero $x = \sum_{i=1}^n c_i x_i$ with $c_i \in K$.*

Remark 8.5. If M is finite dimensional, it suffices to take a single field extension K with K algebraically closed. In general, it suffices to take a field extension of transcendence degree at least $n - 1$ over the algebraic closure $\bar{k}$ of k.

Indeed, it is precisely the necessity of field extensions in this general version of Dade's Lemma which makes field extensions necessary in Theorem 8.1, whereas they are not necessary for Chouinard's theorem (Theorem 1.3). This is because our generality includes the case of a restricted Lie algebra. If we take for example an abelian Lie algebra with trivial restriction map, then Theorem 8.1 becomes precisely Dade's Lemma (see the discussion preceding Corollary 7.3).

Theorem 8.1 says that projectivity for G or for the corresponding Hopf algebra $k[G]^*$ can be detected on subgroups of the from $\mathcal{E}_{r,s} = \mathbb{G}_{a(r)} \times E_s$. Denote the corresponding Hopf algebra $k[\mathcal{E}_{r,s}]^* \simeq k[\mathbb{G}_{a(r)}]^* \otimes kE_s$ by $A_{r,s}$. As an algebra, we have already observed that $A_{r,s}$ is simply isomorphic to $k[x_1, x_2, \dots, x_n]/(x_1^p, x_2^p, \dots, x_n^p)$ where $n = r + s$. Hence, combining Theorem 8.1 with Dade's Lemma, we get the following detection result.

Corollary 8.6. *Let k be a field of characteristic $p > 0$, G be a finite group scheme over k, and M be a rational G-module. Suppose further that the (infinitesimal) connected component of the identity, G_0, of G is unipotent and that all the points of G are k-rational. If for every field extension K/k and subalgebra of the form $K[x]/(x^p) \subset K[G \otimes_k K]^*$, $M \otimes_k K$ is free upon restriction to $K[x]/(x^p)$, then M is projective over G.*

Remark 8.7. Evidently it suffices to take only some of the subalgebras of the form $K[x]/(x^p)$, i.e., those which are "contained in" an elementary subgroup scheme of G.

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